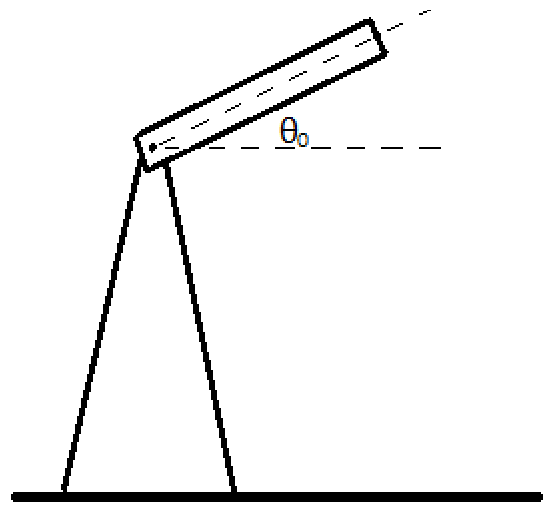
Question:



A 2.0-meter long solid rod is free to rotate about one end which has been fixed to a pivot as shown at right. It is raised to an angle of $\theta_0 = 40°$ above the horizontal and released. What is its angular speed at the moment its position is horizontal?

Answer:

It is assumed there are no non-conservative forces, so mechanical energy will be conserved during the rotation. We set the initial potential energy equal to the final (rotational) kinetic energy.

$$Mgy_{\rm cm} = \frac{1}{2}I\omega^2 \tag{1}$$

The center of mass is located at $y_{\rm cm} = \dfrac{L}{2}\sin\theta_0$ and the moment of inertia of a rod rotating around one end is $I = \dfrac{1}{3}ML^2$. Substituting these into Equation 1 gives

$$\frac{1}{2}MgL\sin\theta_0 = \frac{1}{6}ML^2\omega^2.$$

Solving for angular speed, we have

$$\omega = \sqrt{\frac{3g\sin\theta_0}{L}} \approx \sqrt{\frac{3(9.81\,{\rm m/s^2})(0.643)}{2\,{\rm m}}} \approx \boxed{3.1\,{\rm rad/s}}$$

Numerical solution:
This problem can also be solved dynamically by using Newton's second law in angular coordinates.

$$\sum \tau = I\alpha \tag{2}$$

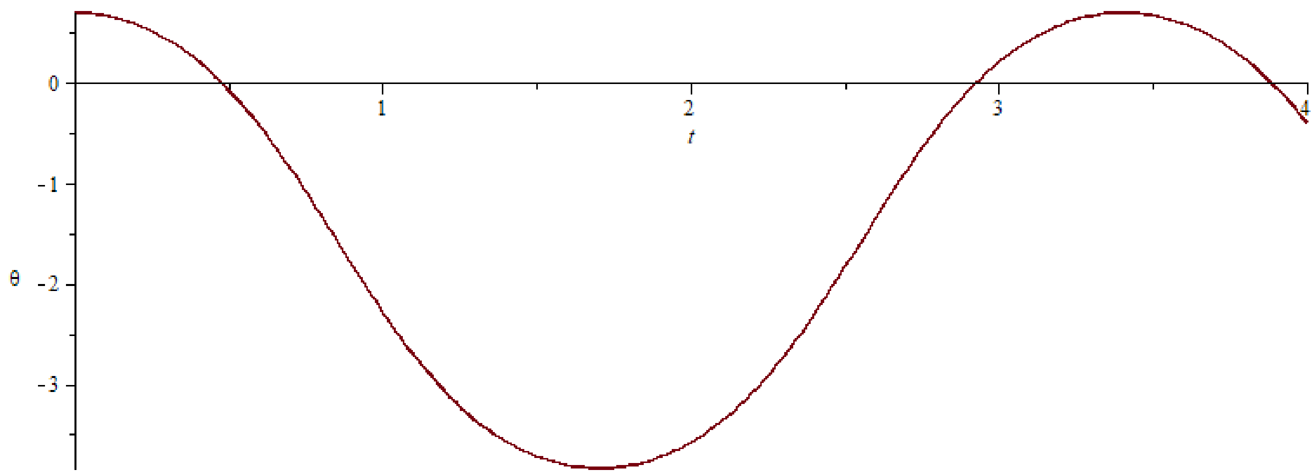
Since the weight of the rod can be considered as acting on the center of mass, we can write

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F} = -\frac{L}{2}\cdot Mg\cdot\sin(\theta + 90°) = -\frac{1}{2}MgL\cos\theta$$

Substituting this and the moment of inertia expression into Equation 2 gives (after some simplification)

$$\frac{{\rm d}^2\theta}{{\rm d}t^2} = -\frac{3g}{2L}\cos\theta$$

This differential equation cannot be solved in terms of elementary functions. However, we can use software to generate a numerical solution, shown below



In particular, we see that the rod reaches a horizontal position at $t \approx 0.48\,\mathrm{s}$ and $\left.\frac{\mathrm{d}\theta}{\mathrm{d}t}\right|_{t=0.48} \approx -3.1\,\mathrm{rad/s}$, which agrees with our previous solution.

Note that the period of the rod's oscillations is considerably larger than the

$$T = 2\pi\sqrt{\frac{I}{MgL_{\mathrm{cm}}}} = 2\pi\sqrt{\frac{2L}{3g}} \approx 2.3\,\mathrm{s}$$

computed using the small angle approximation. However, it is possible to compute the period accurately in terms of the initial angle using a Taylor series expansion as follows:

$$T = 2\pi\sqrt{\frac{2L}{3g}}\left(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \frac{173}{737280}\theta_0^6 + \cdots\right) \approx 3.4\,\mathrm{s}$$

For more details, see https://en.wikipedia.org/wiki/Pendulum_(mathematics).